Substitution into the second term of (A-2) gives

$$\lim_{r\to\infty} \int_{-\pi/2}^{\pi/2} \{A_1{}^{\infty}S + m_{\infty} + \tilde{Z}(S)\} \frac{dS}{S^2 - 1} \qquad \text{(A-8)}$$

with $S = r \exp(i\theta)$, and $dS = ir \exp(i\theta)\, d\theta$. In the limit (A-8) reduces to

$$i\pi A_1{}^{\infty}. \qquad \text{(A-9)}$$

Collecting (A-1), (A-2), (A-6), (A-7), and (A-9) and simplifying gives the result

$$Z(1) = \frac{2}{\pi} \int_0^{\infty} \frac{R(\Omega)}{\Omega^2 + 1}\, d\Omega + A_1{}^{\infty} + A_{-1}{}^{0} + 2 \sum_{k=1}^{L} \frac{A_{-1}{}^{k}}{1 + \Omega_k{}^2}. \qquad \text{(A-10)}$$

Substitution of tan (θ) for Ω in (A-10) gives

$$Z(1) = \frac{2}{\pi} \int_0^{\pi/2} R(\theta)\, d\theta + A_1{}^{\infty} + A_{-1}{}^{0} + 2 \sum_{k=1}^{L} \frac{A_{-1}{}^{k}}{1 + \Omega_k{}^2}. \qquad \text{(A-11)}$$

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A Note on the Finite-Element Solution of Exterior-Field Problems

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Abstract—**An approximate closed-form expression corresponding to the energy functional in an infinite exterior region satisfying Laplace's equation is derived for use with the finite-element method. This expression simplifies the treatment of exterior-field problems in numerical calculations. The expression is given in terms of a few numerical matrices and logarithmic functions.**

I. Introduction

A number of problems in electromagnetics can be formulated in terms of an interior region and an exterior region satisfying Laplace's equation with boundary conditions at infinity. Several methods have been developed for the numerical treatment of these problems, including boundary relaxation [1]–[4], [7] and exterior finite-element methods [5], [6]. A common feature of all of these methods is the solution of the problem in terms of a finite, bounded region called a "picture frame" and the use of Green's functions to determine picture-frame boundary conditions.

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There are two competing approaches to the selection of picture-frame regions and the use of Green's function boundary conditions. In one approach, first proposed by Silvester and Hsieh [5], a single picture frame is defined and the energy-functional contribution of the entire region exterior to this picture frame is evaluated and added to the interior-region energy functional. The solution is therefore obtained by considering the field in all space, but by explicitly solving for the field only in the region interior to the picture frame. In the other approach, proposed by McDonald and Wexler [6], two concentric picture frames are defined and the integral equation relating the potentials between the two picture frames is used to specify the boundary conditions on the outer picture frame. Fields outside of the outer picture frame are never considered in the solution process; the integral equation merely provides a relationship between internal field values.

As developed in the references, however, the energy functional in the exterior region is evaluated by using an integral transformation and weighted Gaussian quadrature formulas. The programming requirements of this procedure are relatively difficult and have limited the application of the technique. In this short paper, the value of the exterior-field energy functional is expressed in closed form. The programming requirements of these closed-form expressions are much less than that of the original transformation-quadrature procedure; hence, the availability and utility of exterior-field finite-element solutions is increased.

II. The Exterior-Field Functional

It is shown in [5] that the energy functional corresponding to the exterior of a finite-element mesh embedded in a space where Laplace's equation applies is given by

$$\mathscr{F}_E = \boldsymbol{a}\boldsymbol{R}\boldsymbol{Q}^{-1}\boldsymbol{R}\boldsymbol{a}^T \qquad (1)$$

where $\boldsymbol{a}$ is a row vector of potential coefficients on the edge of the finite-element mesh and $\boldsymbol{R}$ and $\boldsymbol{Q}$ are the symmetric matrices

$$\boldsymbol{R} = \oint \boldsymbol{\beta}^T(x)\boldsymbol{\beta}(x)\, dx \qquad (2)$$

$$\boldsymbol{Q} = \oint\oint \boldsymbol{\beta}^T(x)G(x,\xi)\boldsymbol{\beta}(\xi)\, d\xi\, dx. \qquad (3)$$

In these equations, $\boldsymbol{\beta}(x)$ is a row vector composed of the interpolation polynomials corresponding to the coefficients $\boldsymbol{a}$ and

$$G(x,\xi) = \frac{1}{2\pi\varepsilon} \ln |x - \xi| \qquad (4)$$

where $|x - \xi|$ indicates the distance between point x and point ξ.

The matrices $\boldsymbol{R}$ and $\boldsymbol{Q}$ in (2) and (3) may be converted into finite-element form by noting that

$$\boldsymbol{\beta}(x) = \sum_{h=1}^{N} \boldsymbol{\beta}^{(h)}(x) \qquad (5)$$

where $\boldsymbol{\beta}^{(h)}(x)$ is a row vector containing the interpolation polynomials for element h ($\boldsymbol{\beta}^{(h)}(x) = 0$ if x is outside element h) and N is the number of elements on the boundary. By making the substitution $z = x/L_h$ where L_h is the length of the exterior side of element h, the interpolation polynomials $\boldsymbol{\beta}^{(h)}(x)$ may be written as

$$\boldsymbol{\beta}^{(h)}(zL_h) = \boldsymbol{p}(z)\boldsymbol{\Gamma} \qquad (6)$$

TABLE I

Values of the matrix Γ for n=1,...,6. Each matrix should be divided by the common denominator $C_D^{(n)}$.

n=1 ; $C_D^{(1)}$= 1

1	0
-1	1

n=2 ; $C_D^{(2)}$= 1

1	0	0
-3	4	-1
2	-4	2

n=3 ; $C_D^{(3)}$= 2

2	0	0	0
-11	18	-9	2
18	-45	36	-9
-9	27	-27	9

n=4 ; $C_D^{(4)}$ = 3

3	0	0	0	0
-25	48	-36	16	-3
70	-208	228	-112	22
-80	288	-384	224	-48
32	-128	192	-128	32

n=5 ; $C_D^{(5)}$ = 24

24	0	0	0	0	0
-274	600	-600	400	-150	24
1125	-3850	5350	-3900	1525	-250
-2125	8875	-14750	12250	-5125	875
1875	-8750	16250	-15000	6875	-1250
-625	3125	-6250	6250	-3125	625

n=6 ; $C_D^{(6)}$ = 10

10	0	0	0	0	0	0
-147	360	-450	400	-225	72	-10
812	-3132	5265	-5080	2970	-972	137
-2205	10440	-20745	22320	-13815	4680	-675
3150	-16740	36990	-43560	28890	-10260	1530
-2268	12960	-30780	38880	-27540	10368	-1620
648	-3888	9720	-12960	9720	-3888	648

where for an nth-order finite element $p(z)$ is the vector

$$p(z) = [1\ z\ z^2 \cdots z^n] \tag{7}$$

and Γ is an $(n+1)$ by $(n+1)$ matrix of constants. In finite-element analysis, the polynomials $\beta^{(h)}(x)$ must equal the equispaced Lagrange interpolation polynomials [8]. For $n = 1, \cdots, 6$, these are obtained by setting the matrix Γ equal to the numerical values given in Table I.

Substituting (5) and (6) into (2) and (3) yields

$$R = \sum_{n=1}^{N} L_n \Gamma^T A \Gamma \tag{8}$$

$$Q = \sum_{g=1}^{N} \sum_{h=1}^{N} \frac{L_g L_h}{2\pi\varepsilon} \Gamma^T B \Gamma \tag{9}$$

where

$$A = \int_0^1 p^T(z) p(z)\, dz \tag{10}$$

$$B = \int_0^1 p^T(z) I\left(\frac{L_h}{L_g} z\right) dz \tag{11}$$

and

$$I(z) = \int_0^1 p(\zeta) \ln (L_g |z - \zeta|)\, d\zeta. \tag{12}$$

The integration of the matrix A is easy. The ith element of the vector $p(z)$ is z^{i-1}; therefore, A is the Hilbert segment matrix [8, p. 196]

$$A_{ij} = \frac{1}{i + j - 1}. \tag{13}$$

III. Evaluation of the Mutual Terms

The evaluation of the integrals for terms involving two different finite elements is made difficult by the fact that the algebraic form of the distance between points z and ζ on different sides of a two-dimensional mesh is very complicated. In order to simplify the expressions involved, the following geometrical approximation is introduced: It is assumed that the distance $|z - \zeta|$ varies as a linear function of ζ

$$|z - \zeta| = (r_2 - r_1)\zeta + r_1 \tag{14}$$

where r_1 and r_2 represent the distances from the point z to the

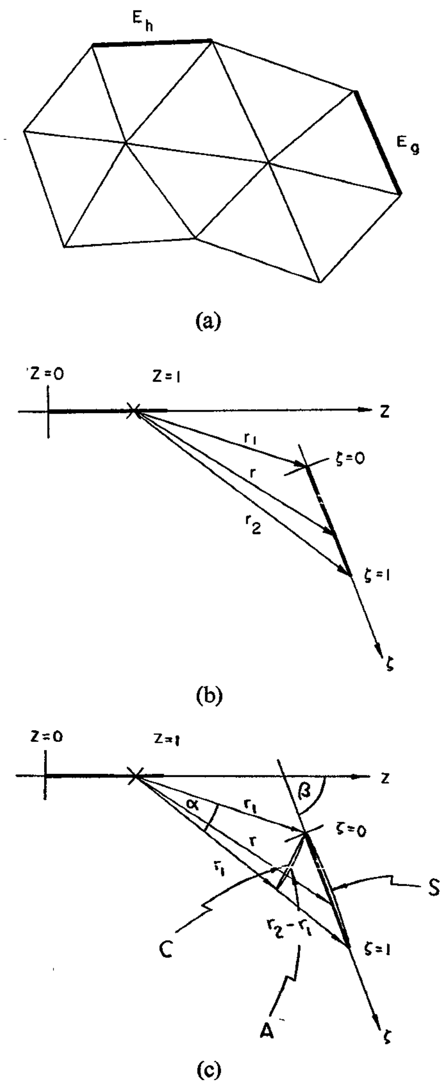

Fig. 1. (a) A typical finite-element subdivision of an unbounded region indicating the locations of the exterior sides used in the integration. (b) The representation of two exterior sides in a finite-element mesh and basis of the geometrical approximation used in the evaluation of $I(z)$. (c) A geometrical interpretation of the errro in the distance $r = |z - \zeta|$ obtained from (14).

two endpoints of the second line segment (see Fig. 1). (Note that r_1 and r_2 are functions of z.)

The error introduced by the approximation in (14) may be determined from Fig. 1(c). In this figure, A is the arc of a circle of radius r_1 centered at the point $(z,0)$ and C is a chord drawn between the endpoints of the arc. The quantity $|z - \zeta|$ in (14) defines another curve S which is obtained by adding a proportionate amount of the length $r_2 - r_1$ to the radius vector r_1 as ζ varies from zero to one. It follows that the distance obtained from (14) is larger than the true distance by an amount equal to the difference between the arc A and the associated chord C. Provided that the angle α subtended by the second element E_g is small, very little error is introduced. The arc angle α will be small provided that either 1) the distance r_1 is much larger than the length of the element E_g, or 2) the angle β between the sides of elements E_h and E_g is small. These two conditions imply that the exterior boundary should be convex and should consist of short sides E_g compared with the overall diameter of the region.

Of course, the error introduced in the evaluation of the exterior functional $\mathscr{F}_E$ by the approximation (14) is much smaller than the error introduced in the distance $|z - \zeta|$ itself since potential at point z is a logarithmic function of this distance.

Using (14) and making the substitution

$$\eta = \zeta + r_1/L_d \tag{15}$$

where $L_d = r_2 - r_1$, the integral $I(z)$ becomes

$$I(z) = \int_{r_1/L_d}^{r_2/L_d} p\left(\eta - \frac{r_1}{L_d}\right) \ln\,(L_g L_d \eta)\, d\eta. \tag{16}$$

Using the binomial theorem, the vector $p(\eta - r_1/L_d)$ becomes

$$p(\eta - r_1/L_d) = p(\eta) M^{+}(-r_1/L_d) \tag{17}$$

where $M^{\pm}(x)$ is an upper triangular matrix possessing the elements

$$M_{ij}^{\pm}(x) = \begin{cases} (\pm 1)^{i-1} \binom{j-1}{i-1} x^{j-i}, & \text{if } i \leq j \\ 0, & \text{if } i > j. \end{cases} \tag{18}$$

Therefore, (16) becomes

$$I(z) = J^{+}(r_1/L_d, r_2/L_d, L_g L_d) M^{+}(-r_1/L_d) \quad (19)$$

where

$$J^{\pm}(a,b,c) = \int_a^b p(\pm\eta) \ln(c\eta)\, d\eta. \quad (20)$$

Since the kth component of $p(\eta)$ is η^{k-1}, the kth component of $J^{\pm}$ reduces to

$$J_k^{\pm}(a,b,c) = \frac{(\pm 1)^{k-1}}{k} \cdot \left\{ b^k \ln(cb) - a^k \ln(ca) - \frac{b^k}{k} + \frac{a^k}{k} \right\}. \quad (21)$$

Furthermore, from (18), since $M_{kj}^{+} = 0$ if $k > j$, the jth component of $I(z)$ is given by

$$I_j(z) = \sum_{k=1}^{j} \frac{(-1)^{j-k}}{k} \binom{j-1}{k-1} \left(\frac{r_1}{L_d}\right)^j \left\{ \left(\frac{r_2}{r_1}\right)^k \ln(L_g r_2) - \ln(L_g r_1) - \frac{1}{k}\left(\frac{r_2}{r_1}\right)^k + \frac{1}{k} \right\}. \quad (22)$$

According to (22), the function $I[(L_h/L_g)z]$ in (11) may be evaluated at any point z. If it is evaluated at the nodes z_i of the interpolation polynomials $\beta(zL_h)$, the following interpolatory approximation results

$$I\left(\frac{L_h}{L_g} z\right) = \beta(z) V \quad (23)$$

where V is a matrix with the elements

$$V_{ij} = I_j(z_i). \quad (24)$$

From (23) and (6), the matrix B becomes

$$B = A\Gamma V \quad (25)$$

where the elements of the matrices A and Γ were previously defined.

IV. Evaluation of the Self Term

The previous equations are modified when evaluating integrals involving only one side of a single finite element. In this case, substituting $\eta = z - \zeta$ in (12) yields

$$I(z) = -\int_z^{z-1} p(z-\eta) \ln(L|\eta|)\, d\eta \quad (26)$$

where L is the length of the element edge. Using (18), (26) becomes

$$I(z) = [J^{-}(0, 1-z, L) + J^{+}(0,z,L)] M^{-}(z). \quad (27)$$

Since $\text{limit}_{z\to 0}\, z^k \ln z = 0$ for $k \geq 1$, the jth component of $I(z)$ is given by

$$I_j(z) = \sum_{k=1}^{j} [J_k^{-}(0, 1-z, L) + J_k^{+}(0,z,L)] M_k^{-}(z)$$
$$= \sum_{k=1}^{j} \frac{1}{k}\binom{j-1}{k-1} \left\{ z^{j-k}(1-z)^k \left[\ln L(1-z) - \frac{1}{k}\right] + (-1)^{k-1} z^j \left[\ln Lz - \frac{1}{k}\right] \right\}. \quad (28)$$

Inserting $I_j(z)$ from (28) into (11) yields for the components of B the integrals

$$B_{ij} = \sum_{k=1}^{j} \frac{1}{k}\binom{j-1}{k-1} \left\{ \int_0^1 z^{i+j-k-1}(1-z)^k \cdot \left[\ln L(1-z) - \frac{1}{k}\right] dz + (-1)^{k-1} \int_0^1 z^{i+j-1} \left[\ln Lz - \frac{1}{k}\right] dz \right\}. \quad (29)$$

When these integrals are evaluated, matrix B may be written in the form

$$B = \Theta \ln(L) - \Phi \quad (30)$$

where Θ and Φ are constant numerical matrices. There are three different expressions for the elements of Θ and Φ: for $i \neq 1$, the substitution $w = 1 - z$ in the first integral, together with the binomial theorem, yields

$$\Theta_{ij} = \frac{(j-1)!}{i+j} \left[\sum_{k=1}^{j} \frac{(-1)^{k-1}}{k!\,(j-k)!} + \sum_{k=1}^{j} \frac{(i+j-k-1)!}{k!\,(j-k)!} \sum_{l=1}^{i+j-k-1} \frac{(-1)^l}{l!\,(k+l+1)(i+j-k-l-1)!} \right] \quad (31)$$

$$\Phi_{ij} = \frac{(j-1)!}{(i+j)^2} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{k!\,(j-k)!} + \frac{(j-1)!}{i+j} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{kk!\,(j-k)!} + (j-1)! \sum_{k=1}^{j} \frac{(i+j-k-1)!}{k!\,(j-k)!} \cdot \sum_{l=1}^{i+j-k-1} \frac{(-1)^l (2k+l+1)}{k(k+l+1)2l!\,(i+j-k-l-1)!}. \quad (32)$$

For $i = 1, j \neq 1$, one obtains

$$\Theta_{1j} = \frac{1}{j(j+1)} + (j-1)! \sum_{k=1}^{j-1} \frac{1}{k!} \cdot \sum_{l=1}^{j-k} \frac{(-1)^l}{(k+l+1)l!\,(j-k-l)!} + \frac{(j-1)!}{j+1} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{k!\,(j-k)!} \quad (33)$$

$$\Phi_{1j} = \frac{2j+1}{j^2(j-1)^2} + (j-1)! \sum_{k=1}^{j-1} \frac{1}{kk!} \cdot \sum_{l=1}^{j-k} \frac{(-1)^l(2k+l+1)}{l!\,(j-k-l)!\,(k+l+1)^2} + \frac{(j-1)!}{(j+1)^2} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{k!\,(j-k)!} + \frac{(j-1)!}{j+1} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{kk!\,(j-k)!} \quad (34)$$

and for $i = j = 1$

$$\Theta_{11} = 1 \quad (35)$$

$$\Phi_{11} = \tfrac{3}{2}. \quad (36)$$

The numerical values of Θ and Φ are given in Tables II and III for polynomial elements of up to order 6; note that since Θ_{ij} and Φ_{ij} are independent of the order of the interpolation

TABLE II

Values of the sixth order Θ matrix; lower order Θ matrices are embedded in these values.

1	0	-1/2	-1	-5/3	-8/3	-17/4
2/9	-1/96	-29/300	-29/180	-57/245	-883/2688	-2123/4536
7/48	-1/50	-19/216	-421/2940	-199/960	-1003/3402	-3557/8400
11/100	-1/48	-23/294	-163/1280	-301/1620	-2239/8400	-2615/6776
4/25	-29/1470	-67/960	-37/324	-21/125	-6173/25410	-3571/10080
11/147	-7/384	-61/972	-31/300	-278/1815	-1349/6048	-9293/28392
29/448	-19/1134	-239/4200	-1597/16940	-709/5040	-209/1014	-1669/5488

TABLE III

Values of the sixth order Φ matrix; lower order Φ matrices are embedded in these values.

3/2	10/9	-19/48	-551/600	-67/48	-19449/9800	-18797/6720
0	-65/144	-37/48	-7937/7200	-2723/1800	-291953/141120	-403943/141120
-19/72	-7/12	-41/48	-4181/3600	-2243/1440	-825/392	-2040539/705600
-19/48	-21/32	-65/72	-11491/9600	-5707/3600	-750389/352800	-19559/6720
-853/1800	-158/225	-2243/2400	-293/240	-14437/9000	-188957/88200	-2062843/705600
-21/40	-2113/2880	-689/720	-2971/2400	-364/225	-28493/13230	-190738229/65029580
-4397/7840	-556/735	-171721/176400	-73543/58800	-71801/44100	-5449/2520	-1166876783/427196860

polynomials, lower order values of Θ and Φ are embedded in the sixth-order matrices.

Finally, combining (25) and (30), one obtains the complete expression for the external energy functional including both mutual and self terms

$$B = A\Gamma V + \Theta \ln (L) - \Phi \tag{37}$$

where all matrices are assumed to be numbered consistently.

V. Conclusions

The exterior-field finite-element functional may be expressed in a convenient closed form requiring the evaluation of a few logarithms and a few operations with small matrices. This closed-form expression is intuitively appealing and simplifies the treatment of boundaries at infinity for regions satisfying Laplace's equations.

In practice, the evaluation of the matrix V in (24) may be simplified for nonadjacent finite-element edges. Provided that the distances r and r_2 are large compared to the edge length L_h, the integrals $I(z)$ are nearly constant for the range of values encountered and need only be computed once (preferably at the midpoint of element h).

The extension of the analysis in this short paper to the solution of exterior-field problems involving wave propagation is not straightforward. The energy associated with the Helmholtz equation in an infinite region is not finite and, as a result, the interior Green's function approach of [6] must be used. In this case, however, there is no exterior S matrix contribution, only an additional set of conditions on the picture-frame nodes.

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Hall Field and Magnetoresistance Effects in Rectangular Waveguide Completely Filled with Semiconductor

J. B. NESS AND M. W. GUNN

***Abstract*—Microwave propagation through a rectangular waveguide completely filled with semiconductor and subject to a transverse magnetic field is analyzed. When the magnetic field is parallel to the broad wall of the waveguide (the x axis), propagation is analyzed in terms of the Hall effect. For the magnetic field parallel to the y axis, the effect of the field on the propagation is shown to be due to longitudinal magnetoresistance effects. Good agreement is obtained between theory and experiment in both cases. The experiments were performed at 30 GHz using n-type germanium.**

Introduction

In this short paper the effects of the application of transverse magnetic fields on the propagation coefficient of a rectangular waveguide completely filled with semiconductor (Fig. 1) are considered theoretically and experimentally. In the absence of the magnetic field the dominant propagation mode in the semiconductor-filled waveguide will be the TE_{10} mode. However, the semiconductor permittivity becomes a tensor in the presence of a magnetic field and the tensor permittivity causes coupling of higher order modes to the TE_{10} mode. It has been shown previously [1] that propagation in the presence of a magnetic field in the x direction is by TE_{0n} modes or by anomalous modes having all six field components.

The method of analysis used here is an approximation technique based on Schelkunoff's "generalized Telegraphists Equation" and adopted previously to analyze the partially filled guide [2]. The propagation characteristics are analyzed in terms of coupling between modes which implies that the propagation mode in the presence of a magnetic field along the x axis is by an anomalous mode rather than TE_{0n} modes.

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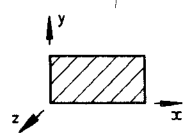

Fig. 1. Fully filled guide.

Since the tensor permittivity is derived assuming that the semiconductor has spherical constant energy surfaces, this method predicts that propagation will be unaffected by a magnetic field directed along the y axis, as the magnetic field is now parallel to the electric field of the TE_{10} mode. However, experimentally the germanium samples do show a magnetic-field dependence for the field along the y axis and the affect is explained qualitatively and quantitatively by the longitudinal magnetoresistance effect.

Experiments performed at 30 GHz using a microwave transmission bridge are used to verify the theoretical analyses for both directions of magnetic field. At this frequency the results indicate that the effects of carrier inertia are measurable, and by taking the relaxation time into consideration better agreement between theory and experiment is obtained.

Theory

Consider an electromagnetic wave propagating through a semiconductor. The total current density can be written as

$$\boldsymbol{J} = \boldsymbol{J}_d + \boldsymbol{J}_c \tag{1}$$

where

$\boldsymbol{J}_d$ the displacement current density;
$\boldsymbol{J}_c$ the conduction current density.

In the presence of a magnetic field the semiconductor permittivity becomes a tensor so that the current density can now be written as

$$\boldsymbol{J} = [\varepsilon] \frac{\partial \boldsymbol{E}}{\partial t}. \tag{2}$$

The particular form of the permittivity tensor will depend on the assumptions used in the derivation. Engineer and Nag [1] have developed a form for this tensor by including the Hall field in Maxwell's equations, although all terms were assumed frequency independent. Kataoka and Fujisada [3] have obtained expressions for the tensor permittivity terms using the basic equations of motion although the lattice permittivity term was neglected in this derivation. The following derivation based on the equation of motion for electrons under the influence of an applied alternating electric field and a steady magnetic field [4], includes both of these factors. Thus

$$m_e{}^* \frac{d\bar{v}_e}{dt} + m_e{}^* \frac{\bar{v}_e}{\tau_e} = -q(\boldsymbol{E}e^{j\omega t} + \bar{v}_e \times \boldsymbol{B}) \tag{3}$$

where

$\boldsymbol{B}$ the magnetic-flux density;
τ_e the relaxation time for electrons assumed isotropic and constant;
$\bar{v}_e$ the average induced velocity of the electrons;
$m_e{}^*$ the effective mass of the electrons.

For electromagnetic propagation along the z direction and with the magnetic field in the x direction, (3) can be written in